Probability Hypothesis Density Filtering with Sensor Networks and Irregular Measurement Sequences*

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Abstract – *The problem of multi-object tracking with sensor networks is studied using the probability hypothesis density filter. The sensors are assumed to generate signals which are sent to an estimator via parallel channels which incur independent delays. These signals may arrive out-of-order (out-of-sequence), be corrupted or even lost due to, e.g., noise in the communication medium and protocol malfunctions. In addition, there may be periods when the estimator receives no information. A closed-form, recursive solution to the considered problem is detailed that generalizes the Gaussian-mixture probability hypothesis density (GM-PHD) filter previously detailed in the literature.*

Keywords: PHD filtering; random-set-based estimation; irregular measurement sequences; out-of-sequence measurements; sensor networks; delay-tolerant PHD filtering.

1 Introduction

Suppose the observation set at each time step is a collection of indistinguishable partial observations, only some of which are generated by targets. Each observation *vector* in the observation set that is generated by a true target object can be regarded as a *point* in some observation space. In addition, a number of points in the observation set might be spurious (generated by no true target). The objective of multi-object tracking is to jointly estimate, at each time step, the number of targets and their states from this sequence of noisy and cluttered observation sets (involving missed detections at some time steps). This problem, at the conceptual level, is not new and the reader may refer to [1–4] for an overview. A theoretically optimal, Bayesian filtering, framework can be formulated for the multi-sensor/multi-target tracking problem using the concept of finite set statistics (FISST) developed in [3]. Although it is conceptually possible to write down the complete Bayesian solution to the multi-object tracking problem within a random set framework, the implementation complexity makes the algorithm impractical.

Mahler [5] thus proposed a first-oder moment approximation to the full Bayesian solution and termed the first-moment, the probability hypothesis density (PHD). The PHD recursion involves multiple integrals that have no closed form solutions in general. A generic sequential Monte Carlo implementation [6–8] has been proposed and accompanied by convergence results [8–10]. In these approaches, state estimates are extracted from the particles representing the posterior intensity using clustering techniques. Alternatively, an analytic solution to the PHD recursion was presented in [11] for problems involving linear Gaussian target dynamics, a Gaussian birth model and linear Gaussian (partial) observations. This novel solution is analogous to the Kalman filter as a solution to the single-target Bayes filter. It is shown in [11] that when the initial prior intensity is a Gaussian mixture, the posterior intensity at any subsequent time step is also a Gaussian mixture. Furthermore, it is shown in [12] that the Gaussian-mixture PHD recursions can approximate the true posterior intensity to any desired degree of accuracy. See [3,4,7] for a comprehensive background on random finite set-based target tracking and the PHD filter.

A standard assumption in classical filtering theory [13] is that the observations are available either immediately or with a constant/known delay. However, in many applications, involving networked sensors [14], the observations are transmitted to the estimator via communication channels with considerable, irregular, and a priori unknown delays [15–17]. The problem considered in this paper is one of multi-object tracking in adverse environments via sensor networks which send observations to the estimator over unreliable communication channels.

1.0.1 Original Contribution

In this paper, we study the problem of multiple-sensor-based multi-object tracking in adverse environments involv-

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ing clutter (false positives), missing measurements (false negatives) and random target births and deaths (a priori unknown target numbers). The sensors generate signals which are sent to the estimator via parallel channels which incur independent delays. These signals may arrive out of order, be corrupted or even lost due to, e.g., noise in the communication medium and protocol malfunctions. In addition, there may be periods when the estimator receives no information. The estimation problem is then said to involve irregular measurement sequences.

Following [17], we assume that neither the delays in the communication channels nor their statistics are known in advance. However, each signal message is marked with a "time stamp" indicating the time at which the sensor generated the signal (or potentially the time at which the sensor transmits the message). In addition, signal messages received at the estimator can be correctly associated to individual sensor sources (albeit this last type of association can be relaxed in the random set framework discussed in this paper).

A closed-form, recursive solution to the considered problem is detailed that generalizes the Gaussian-mixture probability hypothesis density (GM-PHD) filter previously detailed in the literature [11, 12]. This generalization allows the GM-PHD framework to be applied in more realistic network scenarios involving delays and irregular measurement sequences where particular measurements can arrive out of order with respect to the generating sensor and also with respect to the signals generated by the other sensors in the network.

1.1 Organization

The paper is organized as follows. In Sections II through IV the standard random-set-based tracking problem with multiple-sensors is outlined. Specifically, in Section II the conceptual models for the target dynamics and the multi-sensor measurement system are introduced within the framework of random finite sets. Also, the complete recursive, Bayesian, estimation algorithm is outlined for completeness. In Section III the first-order moment (the probability hypothesis density (PHD)) approximation to the full Bayesian filter is outlined. In Section IV, the Gaussian-mixture-based implementation of the PHD filter (the GM-PHD filter [11]) is presented for completeness. The new contribution of this paper appears in Section V. In particular, a new algorithm is outlined in section V that extends the GM-PHD filter to permit irregular measurement sequences from multi-sensor networks with unreliable communication channels. A conclusion is given in Section VI.

2 A Conceptual Model with Standard Measurement Sequences

The state of a single target i is represented by the random variable $\mathbf{X}_t^i$ measured on the space $\mathcal{E} \subseteq \mathbb{R}^{n_x}$ and is assumed to obey

$$\mathbf{X}_t^i = \mathbf{\Phi}_t^i \mathbf{x}_{t-1}^i + \mathbf{W}_t^i \tag{1}$$

where the input $\{\mathbf{W}_t^i\}$ is a sequence of independent Gaussian random variables. Note that the transition density for a single target is now given by

$$f_{t|t-1}^i(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i) = \mathcal{N}(\mathbf{x}_t^i;\ \mathbf{\Phi}_t^i \mathbf{x}_{t-1}^i, \mathbf{\Sigma}_t) \tag{2}$$

where $\mathcal{N}(\cdot;\ \mathbf{m}, \mathbf{P})$ denotes a Gaussian density function with mean $\mathbf{m}$ and covariance $\mathbf{P}$ and $\mathbf{\Sigma}_t^i$ is the covariance $\mathbf{W}_t^i$. We assume throughout that $f_{t|t-1}^i = f_{t|t-1}$ Now consider a number n_s of sensors located in $\mathbb{R}^d$. Each sensor j observes some function

$$\mathbf{Z}_t^{(j,i)} = \mathbf{\Gamma}_t^j \mathbf{x}_t^i + \mathbf{V}_t^j \tag{3}$$

in the observation space $\mathcal{M} \subseteq \mathbb{R}^{n_{z_j}}$ where typically $n_{z_j} \leq n_x$. Note that certain measurement spaces, such as target bearing measurements, can be approximated by subspaces of the real line. The input $\{\mathbf{V}_t^j\}$ is a sequence of independent Gaussian random variables.The measurement likelihood function is

$$g_t^j(\mathbf{z}_t^j|\mathbf{x}_t^i) = \mathcal{N}(\mathbf{z}_t^j;\ \mathbf{\Gamma}_t^j \mathbf{x}_t^i, \mathbf{\Lambda}_t^j) \tag{4}$$

where $\mathbf{\Lambda}_t^j$ is the covariance of $\mathbf{V}_t^j$. We assume that $g_t^j = g_t$.

Now consider a multiple target scenario and let $\mathbf{X}_t^i$ denote the random vector associated with the state of the i^{th} target. The number of targets is time-varying and given by N_t and similarly the number of measurements received is time-varying and given by $M_t = \sum_{i=1}^{n_s} M_t^i$ where M_t^i is the number of measurements received at the i^{th} sensor.

The state evolution considered in this paper incorporates target motion, birth and death[1]. The probability that any target i continues to exist at time t given that it exists at $t-1$ is given by the constant survival probability p_S. The targets born at time t are characterized by the Poisson random finite set $\mathfrak{B}_t$ with intensity

$$\beta_t = \sum_{i=1}^{J_t^\beta} w_t^{(\beta,i)} \mathcal{N}(\mathbf{x};\ \mathbf{m}_t^{(\beta,i)}, \mathbf{P}_t^{(\beta,i)}) \tag{5}$$

Now it follows that

$$\mathfrak{X}_t = \left[\bigcup_{\mathbf{X}_{t-1}^i \in \mathfrak{X}_{t-1}} \mathfrak{S}_{t|t-1}(\mathbf{X}^i) \right] \cup \mathfrak{B}_t \tag{6}$$

where

$$\mathfrak{S}_{t|t-1}(\mathbf{X}_{t-1}^i) = \begin{cases} \mathbf{X}_t^i \cap \mathcal{E} & \text{with probability } p_S \\ \emptyset & \text{with probability } 1 - p_S \end{cases} \tag{7}$$

and where the evolution of $\mathbf{X}_{t-1}^i$ follows (1).

The measurement model considered in this paper incorporates true target measurements, missed measurements and spurious false (clutter) measurements. The probability that any target i is detected by sensor j is given by the constant detection probability p_D^j. The clutter measured at time t by

[1]For brevity, we neglect target spawning but note that it is trivial to include in the subsequent derivations.

sensor j is characterized by a Poisson random finite set $\mathfrak{C}_t^j$ with intensity

$$\kappa_t^j = \gamma_t^j \mathcal{U}(\mathcal{R}_j) \quad (8)$$

where $\mathcal{U}(\mathcal{R}_j)$ denotes a uniform density function over some region of interest $\mathcal{R}_j \subset \mathcal{E}$, e.g. the surveillance region of sensor j. We assume throughout that $\kappa_t^j = \kappa_t$. Now it follows that

$$\mathfrak{Z}_t^j \;=\; \left[\bigcup_{\mathbf{X}_t^i \in \mathfrak{X}_t} \mathfrak{D}_t(\mathbf{X}^i) \right] \cup \mathfrak{C}_t^j \quad (9)$$

where, assuming now that $p_D^j = p_D$, we have

$$\mathfrak{D}_t(\mathbf{X}_t^i) = \begin{cases} \mathbf{Z}_t^j \cap \mathcal{M} & \text{with probability } p_D \\ \emptyset & \text{with probability } 1 - p_D \end{cases} \quad (10)$$

and where $\mathbf{Z}_t^j$ is modeled by (3) and (4) with $g_t^j = g_t$. Now $\mathfrak{Z}_t = \bigcup_{i=1}^{n_s} \mathfrak{Z}_t^i$. More specifically, each element $\mathbf{Z}_t \in \mathfrak{Z}_t^j$ can be represented (in practice) as the pair $(\mathbf{Z}_t, j)$ and in this case the union $\mathfrak{Z}_t = \bigcup_{i=1}^{n_s} \mathfrak{Z}_t^i$ should be interpreted as a disjoint union of random sets.

Now, similarly, to the single-target case, we define a multiple-target transition density $f_{t|t-1}(\mathfrak{X}_t|\mathfrak{X}_{t-1})$ and likelihood $g_t(\mathfrak{Z}_t|\mathfrak{X}_t)$[2]. Using finite set statistics (FISST), we can find explicit expressions for the multiple-target transition density and likelihoods. The measurement likelihood function corresponding to the single-sensor model (9) is given by

$$\begin{aligned} g_t^j(\mathfrak{Z}_t^j|\mathfrak{X}_t) \;&=\; e^{-\gamma_t} \cdot \prod_{\mathbf{z} \in \mathfrak{Z}_t^j} \kappa_t \cdot \prod_{\mathbf{z} \in \mathfrak{Z}_t^j} (1 - p_D) \cdot \\ &\sum_{\theta^j} \prod_{i:\theta^j(i)>0} \frac{p_D g_t(\mathbf{z}_t(\theta^j(i))|\mathbf{x}_t^i)}{\kappa_t(1 - p_D)} \end{aligned} \quad (11)$$

where the summation is taken over all associations $\theta^j : \{1, \ldots, N_t\} \to \{1, \ldots, M_t^j\}$ and where $\mathbf{z}_t(\theta^j(i))$ is an element in $\mathfrak{Z}_t^j$ marked by the function θ^j; see [4]. The multi-sensor likelihood function $g_t(\mathfrak{Z}_t|\mathfrak{X}_t)$ is then given by

$$g_t(\mathfrak{Z}_t|\mathfrak{X}_t) = \prod_{j=1}^{n_s} g_t^j(\mathfrak{Z}_t^j|\mathfrak{X}_t) \quad (12)$$

under the assumptions adopted in (9).

Now the multiple-target transition density $f_{t|t-1}(\mathfrak{X}_t|\mathfrak{X}_{t-1})$ can be similarly found using FISST. Under the model (6) it is given by

$$\begin{aligned} f_{t|t-1}(\mathfrak{X}_t|\mathfrak{X}_{t-1}) &= \prod_{\mathbf{x} \in \mathfrak{X}_t} \beta_t \cdot \prod_{\mathbf{x} \in \mathfrak{X}_{t-1}} (1 - p_S) \cdot \\ e^{-\left(\sum_{i=1}^{J_t^\beta} w_t^{(\beta,i)}\right)} &\cdot \sum_{\theta} \prod_{i:\theta(i)>0} \frac{p_S f_{t|t-1}(\mathbf{x}_t^{\theta(i)}|\mathbf{x}_{t-1}^i)}{\beta_t(1 - p_D)} \end{aligned} \quad (13)$$

[2] In the case of random finite sets we make no distinction between the random sets and their realizations.

where the summation is taken over all associations $\theta : \{1, \ldots, N_{t-1}\} \to \{1, \ldots, N_t\}$; see [4]. Target spawning can be easily included but complicates the notation [4].

Let $p_t(\mathfrak{X}_t|\mathfrak{Z}_{1:t})$ denote the multiple target posterior density. Then, the optimal multiple target Bayes filter propagates the multiple target posterior in time via the recursion

$$\begin{aligned} &p_{t|t-1}(\mathfrak{X}_t|\mathfrak{Z}_{1:t-1}) = \\ &\quad \int f_{t|t-1}(\mathfrak{X}_t|\mathfrak{X}) p_{t-1}(\mathfrak{X}|\mathfrak{Z}_{1:t-1}) \, \mu_S(\mathrm{d}\mathfrak{X}) \end{aligned} \quad (14)$$

$$\begin{aligned} &p_t(\mathfrak{X}_t|\mathfrak{Z}_{1:t}) = \\ &\quad \frac{g_t(\mathfrak{Z}_t|\mathfrak{X}_t) p_{t|t-1}(\mathfrak{X}_t|\mathfrak{Z}_{1:t-1})}{\int g_t(\mathfrak{Z}_t|\mathfrak{X}_t) p_{t|t-1}(\mathfrak{X}_t|\mathfrak{Z}_{1:t-1}) \, \mu_S(\mathrm{d}\mathfrak{X})} \end{aligned} \quad (15)$$

where μ_S is an appropriate reference measure on $\mathcal{F}(\mathcal{E})$. We remark that FISST is the first systematic approach to multi-target filtering that uses random finite sets in the Bayesian framework presented above [4, 11]. The recursive Bayesian filter for multiple target tracking suffers from a severe computational requirement and only a few implementations have been considered [4, 7, 18, 19].

3 A First-Order Moment Approximation: The PHD Filter

The probability hypothesis density (PHD) filter is an approximation developed to alleviate the computational burden of the multi-target Bayes filter. The PHD filter propagates the posterior intensity, a first-order statistical moment of the posterior multi-target state.

Assumption 1. *The predicted multi-target random finite set governed by $p_{t|t-1}$ is Poisson.*

For a RFS $\mathfrak{X}$ on $\mathcal{E}$ with probability distribution P, its first-order moment is a non-negative function v on $\mathcal{E}$, called the intensity, such that for each region $\mathcal{A} \subseteq \mathcal{E}$

$$\int |\mathfrak{X} \cap \mathcal{A}| \; \mathrm{P}(\mathrm{d}\mathfrak{X}) = \int_{\mathcal{A}} v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad (16)$$

where the expected number of targets is

$$\mathrm{E}[N] = \int_{\mathcal{E}} v(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad (17)$$

The local maxima of the intensity v are points in $\mathfrak{X}$ with the highest local concentration of expected number of elements. These maxima can be used to generate estimates for the elements of $\mathfrak{X}$.

Let v_t and $v_{t|t-1}$ denote the intensities associated with the multiple target posterior density p_t and the multiple target predicted density $p_{t|t-1}$. Following [4] we can show that the posterior intensity is $v_t(\mathbf{x}) = v_t^{n_s}(\mathbf{x})$ where

$$\begin{aligned} &v_t^k(\mathbf{x}) = (1 - p_D^k) v_t^{k-1}(\mathbf{x}) + \\ &\sum_{\mathbf{z} \in \mathfrak{Z}_t^k} \frac{p_D^k g_t^{(k,')}(\mathbf{z}|\mathbf{x}) v_t^{k-1}(\mathbf{x})}{\kappa_t^k + p_D^k \int g_t^{(k,')}(\mathbf{z}|\mathbf{x}') v_t^{k-1}(\mathbf{x}') \, \mathrm{d}\mathbf{x}'} \end{aligned} \quad (18)$$

and $v_t^0(\mathbf{x}) = v_{t|t-1}(\mathbf{x})$. The PHD predictor is given by

$$v_{t|t-1}(\mathbf{x}) = b_t\beta_t + p'_S \int f_{t|t-1}(\mathbf{x}|\mathbf{x}')v_{t-1}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \quad (19)$$

Recall that we have neglected spawned targets for simplicity.

4 A Multiple Sensor and Multiple Target Gaussian-Sum PHD Filter

In this section we present the two main results from [11] for completeness[3] and for later comparison with the newly developed algorithm.

Proposition 1 ([11]). *Suppose the modeling assumptions presented hold and that the posterior intensity at time $t-1$ is a Gaussian mixture of the form*

$$v_{t-1}(\mathbf{x}) = \sum_{i=1}^{J_{t-1}} w_{t-1}^i \mathcal{N}(\mathbf{x};\ \mathbf{m}_{t-1}^i, \mathbf{P}_{t-1}^i) \quad (20)$$

Then, the predicted intensity at time t is given by

$$v_{t|t-1}(\mathbf{x}) = \beta_t + p_S \sum_{i=1}^{J_{t-1}} w_{t-1}^i \mathcal{N}(\mathbf{x};\ \mathbf{m}_{t|t-1}^i, \mathbf{P}_{t|t-1}^i) \quad (21)$$

where

$$\mathbf{m}_{t|t-1}^i = \mathbf{\Phi}_t \mathbf{m}_{t-1}^i \quad (22)$$

$$\mathbf{P}_{t|t-1}^i = \mathbf{\Sigma}_t + \mathbf{\Phi}_t \mathbf{P}_{t-1}^i \mathbf{\Phi}_t^\top \quad (23)$$

and is also a Gaussian mixture.

Proposition 2 (Adapted from [11, 20]). *Suppose the modeling assumptions presented hold and that the predicted intensity at time t is a Gaussian mixture of the form*

$$v_{t|t-1}(\mathbf{x}) = \sum_{i=1}^{J_{t|t-1}} w_{t|t-1}^i \mathcal{N}(\mathbf{x};\ \mathbf{m}_{t|t-1}^i, \mathbf{P}_{t|t-1}^i) \quad (24)$$

Then, the posterior intensity at time t is given by $v_t(\mathbf{x}) = v_t^{n_s}(\mathbf{x})$ where

$$v_t^k(\mathbf{x}) = (1-p_D)v_t^{k-1}(\mathbf{x}) + \sum_{\mathbf{z}\in\mathfrak{Z}_t^k} \sum_{i=1}^{J_{t|t-1}} w_{t|t}^{(i,k)}(\mathbf{z})\mathcal{N}(\mathbf{x};\ \mathbf{m}_{t|t}^{(i,k)}(\mathbf{z}), \mathbf{P}_{t|t}^{(i,k)}) \quad (25)$$

$$= \sum_{i=1}^{J_t^k} w_t^{(i,k)} \mathcal{N}(\mathbf{x};\ \mathbf{m}_t^{(i,k)}, \mathbf{P}_t^{(i,k)}) \quad (26)$$

and $v_t^0(\mathbf{x}) = v_{t|t-1}(\mathbf{x})$ and where

$$w_{t|t}^{(i,k)}(\mathbf{z}) = \frac{p_D w_t^{(i,k-1)} q_t^{(i,k)}(\mathbf{z})}{\kappa_t + p_D \sum_{\ell=1}^{J_{t|t-1}^k} w_t^{(\ell,k-1)} q_t^{(\ell,k)}(\mathbf{z})} \quad (27)$$

$$q_t^{(i,k)}(\mathbf{z}) = \mathcal{N}(\mathbf{z}; \mathbf{\Gamma}_t \mathbf{m}_t^{(i,k-1)}, \mathbf{\Lambda}_t + \mathbf{\Gamma}_t \mathbf{P}_t^{(i,k-1)} \mathbf{\Gamma}_t^\top)$$

$$\mathbf{m}_{t|t}^i(\mathbf{z}) = \mathbf{m}_t^{(i,k-1)} + \mathbf{K}_t^{(i,k)}(\mathbf{z} - \mathbf{\Gamma}_t \mathbf{m}_t^{(i,k-1)}) \quad (28)$$

$$\mathbf{K}_t^{(i,k)} = \mathbf{P}_t^{(i,k-1)} \mathbf{\Gamma}_t^\top (\mathbf{\Lambda}_t + \mathbf{\Gamma}_t \mathbf{P}_t^{(i,k-1)} \mathbf{\Gamma}_t^\top)^{-1}$$

$$\mathbf{P}_{t|t}^i = (\mathbf{I} - \mathbf{K}_t^{(i,k)} \mathbf{\Gamma}_t) \mathbf{P}_t^{(i,k-1)} \quad (29)$$

and $v_t(\mathbf{x})$ is also a Gaussian mixture.

The preceding propositions show how the Gaussian components of the posterior intensity are analytically propagated in time [11]. That is, the preceding propositions provide a closed-form recursive solution to the PHD filtering problem[4]. We extend Propositions 1 and 2 in the next section to account for more realistic networked scenarios.

5 The Main Contribution

Recall the random measurement set at the j^{th} sensor is given by

$$\mathfrak{Z}_t^j = \left[\bigcup_{\mathbf{X}_t^i \in \mathfrak{X}_t} \mathfrak{D}_t(\mathbf{X}^i) \right] \cup \mathfrak{C}_t^j \quad (30)$$

where

$$\mathfrak{D}_t(\mathbf{X}_t^i) = \begin{cases} \mathbf{Z}_t^j \cap \mathcal{M} & \text{with probability } p_D \\ \emptyset & \text{with probability } 1-p_D \end{cases} \quad (31)$$

and where $\mathbf{Z}_t^j$ follows (3) and (4) with $g_t^j = g_t$ and $\kappa_t^j = \kappa_t$. The measurement $\mathfrak{Z}_t^j$ produced at a time t is sent to the estimator via a communication channel and arrives at the time $t + \tau^j(t)$. We put $\tau^j(t) \triangleq \infty$ if the signal $\mathfrak{Z}_t^j$ is lost.

Assumption 2. *The delays in the channels are bounded by a known constant $\tau^j(t) \leq \psi$ whenever $\tau^j(t) \neq \infty$.*

Note that if $\mathfrak{Z}_t^j$ arrives at some time later than $t + \psi$ then it is simply discarded. The set of measurements that arrive at the estimator at time t is then given by

$$\mathfrak{Z}_t = \{\mathfrak{Z}_\theta^i\}_{(i,\theta)\in\mathfrak{A}_t} \quad (32)$$

where $\mathfrak{A}_t = \{(i,\theta) : \theta + \tau^i(\theta) = t\}$ and obviously $\mathfrak{Z}_t = \emptyset$ if $\mathfrak{A}_t = \emptyset$. For future brevity we suppose the elements (i,θ) in $\mathfrak{A}_t$ are randomly ordered at time t and indexed by $k = 1, \ldots, |\mathfrak{A}_t|$ so that we re-define $\mathfrak{A}_t = \{(i_k, \theta_k) : \theta + \tau^i(\theta) = t\}$. Note the particular ordering is irrelevant apart from providing a notational simplification.

[3]Note we present a slight modification of Proposition 2 in the spirit of [20] that accounts for multiple sensors.

[4]In order to be computationally feasible the authors of [11] proposed a simple pruning and merging strategy. Error bounds have been established [12] for the pruning and merging stages of the algorithm which ensure that the accuracy of these stages can be controlled.

Remark 1. *We see now that the algorithm proposed in this paper is designed to deal with multiple sensors transmitting signals over an unreliable network. Each sensor's signal can incur an independent random delay and may arrive out-of-sequence with other signals generated by the same sensor and other signals generated by other sensors in the network.*

Now denote by $\breve{v}_{k|t}$ the posterior intensity associated with the multiple target posterior density p_k based on the measurements $\mathfrak{Z}_{1:t}$. We want to estimate the following set of prediction and posterior intensities

$$\begin{aligned} \mathfrak{V}_{t|t-1} &= \{\widehat{v}_{t|t-1}, \widehat{v}_{t-1|t-1}, \ldots, \widehat{v}_{t-\psi-1|t-1}\} \quad (33) \\ \mathfrak{V}_{t} &= \{\breve{v}_{t|t}, \breve{v}_{t-1|t}, \ldots, \breve{v}_{t-\psi|t}\} \quad (34) \end{aligned}$$

within a recursive Gaussian mixture framework. Note that $\breve{v}_{t|t} = v_t$ in the notation of the previous section and note that in the case of no delayed measurements from any sensors, i.e. $\psi = 0$, then it is enough to estimate $\breve{v}_{t|t}$ using the recursions outlined in the previous section.

Theorem 1 (Generalized Prediction)**.** *Suppose the modeling assumptions introduced hold and that the posterior intensities in (34) at time $t-1$ are Gaussian mixtures of the form*

$$\breve{v}_{t-k|t-1}(\mathbf{x}) = \sum_{i=1}^{\breve{J}_{t-k|t-1}} \breve{w}^i_{t-k|t-1} \mathcal{N}(\mathbf{x};\, \breve{\mathbf{m}}^i_{t-k|t-1}, \breve{\mathbf{P}}^i_{t-k|t-1}) \quad (35)$$

for $k = 1, \ldots, \psi + 1$. The predicted intensity functions at time t are given by

$$\widehat{v}_{t-j|t-1}(\mathbf{x}) = \sum_{i=1}^{\widehat{J}_{t-j|t-1}} \widehat{w}^i_{t-j|t-1} \mathcal{N}(\mathbf{x};\, \widehat{\mathbf{m}}^i_{t-j|t-1}, \widehat{\mathbf{P}}^i_{t-j|t-1}) \quad (36)$$

for $j = 1, \ldots, \psi$ and where $\widehat{J}_{t-j|t-1} = \breve{J}_{t-k|t-1}$, $\widehat{w}^i_{t-j|t-1} = \breve{w}^i_{t-k|t-1}$, $\widehat{\mathbf{m}}^i_{t-j|t-1} = \breve{\mathbf{m}}^i_{t-k|t-1}$ and $\widehat{\mathbf{P}}^i_{t-j|t-1} = \breve{\mathbf{P}}^i_{t-k|t-1}$ when $j = k$. Also, we have

$$\begin{aligned} \widehat{v}_{t|t-1} &= \beta_t + p_S \sum_{i=1}^{\breve{J}_{t-1|t-1}} \breve{w}^i_{t-1|t-1} \mathcal{N}(\mathbf{x}; \widetilde{\mathbf{m}}^i_{t|t-1}, \widetilde{\mathbf{P}}^i_{t|t-1}) \\ &\triangleq \sum_{i=1}^{\widehat{J}_{t|t-1}} \widehat{w}^i_{t|t-1} \mathcal{N}(\mathbf{x};\, \widehat{\mathbf{m}}^i_{t|t-1}, \widehat{\mathbf{P}}^i_{t|t-1}) \quad (37) \end{aligned}$$

where

$$\begin{aligned} \widetilde{\mathbf{m}}^i_{t|t-1} &= \mathbf{\Phi}_t \breve{\mathbf{m}}^i_{t-1|t-1} \quad (38) \\ \widetilde{\mathbf{P}}^i_{t|t-1} &= \mathbf{\Sigma}_t + \mathbf{\Phi}_t \breve{\mathbf{P}}^i_{t-1|t-1} \mathbf{\Phi}_t^\top \quad (39) \end{aligned}$$

and $v_{t-j|t-1}$ and $v_{t|t-1}$ are Gaussian-sums for $j = 1, \ldots, \psi$. Suppose also we have the set of covariance matrices $\mathbf{W}^{i,+}_{a,b}$ for all pairs $a, b \in \{0, \ldots, \psi\}$ which are detailed in Theorem 2. The predicted covariance matrices of $\mathbf{W}^{i,-}_{a,b}$ are given by

$$\mathbf{W}^{i,-}_{a,b} = \begin{cases} \mathbf{\Sigma}_t + \mathbf{\Phi}_t \mathbf{W}^{i,+}_{a,b} \mathbf{\Phi}_t^\top & \text{if } a = b = 0 \\ \mathbf{\Phi}_t \mathbf{W}^{i,+}_{a,b-1} & \text{if } a = 0, b > 0 \\ \mathbf{W}^{i,+}_{a-1,b} \mathbf{\Phi}_t^\top & \text{if } a > 0, b = 0 \\ \mathbf{W}^{i,+}_{a-1,b-1} & \text{if } a > 0, b > 0 \end{cases} \quad (40)$$

for all pairs $a, b \in \{0, \ldots, \psi\}$.

The prediction intensity functions are relatively simple in comparison to the update functions presented next. Moreover, it should be clear that if $\psi = 0$ then the prediction algorithm proposed in Theorem 1 collapses to the algorithm proposed in Proposition 1.

Theorem 2 (Generalized Update)**.** *Suppose the modeling assumptions introduced hold and that the prediction intensities at time t are Gaussian mixtures of the form*

$$\widehat{v}_{t-k|t-1}(\mathbf{x}) = \sum_{i=1}^{\widehat{J}_{t-k|t-1}} \widehat{w}^i_{t-k|t-1} \mathcal{N}(\mathbf{x};\, \widehat{\mathbf{m}}^i_{t-k|t-1}, \widehat{\mathbf{P}}^i_{t-k|t-1}) \quad (41)$$

for $k = 0, \ldots, \psi$. Suppose also we have the set of covariance matrices $\mathbf{W}^{i,-}_{a,b}$ for all pairs $a, b \in \{0, \ldots, \psi\}$.

Then, the posterior intensity function $\breve{v}_{t|t} = \breve{v}^{|\mathfrak{A}_t|}_{t|t}$ at time t is defined recursively by

$$\begin{aligned} \breve{v}^s_{t-k|t} &= (1 - p_D)\breve{v}^{s-1}_{t-k|t}(\mathbf{x}) + \\ &\sum_{\mathbf{z} \in \mathfrak{Z}^{j_s}_{\theta_s}} \sum_{i=1}^{\widehat{J}_{t-k|t-1}} w^{(i,s)}_{t-k|t}(\mathbf{z}) \mathcal{N}(\mathbf{x};\, \mathbf{m}^{(i,s)}_{t-k|t}(\mathbf{z}), \mathbf{P}^{(i,s)}_{t-k|t}) \quad (42) \\ &\triangleq \sum_{i=1}^{\breve{J}^s_{t-k|t}} \breve{w}^{(i,s)}_{t-k|t} \mathcal{N}(\mathbf{x};\, \breve{\mathbf{m}}^{(i,s)}_{t-k|t}, \breve{\mathbf{P}}^{(i,s)}_{t-k|t}) \quad (43) \end{aligned}$$

and $\breve{v}^0_{t|t} = \widehat{v}_{t|t-1}(\mathbf{x})$ with $s = 1, \ldots, |\mathfrak{A}_t|$. We define $\mathbf{W}^{(i,0),-}_{a,b} = \mathbf{W}^{i,-}_{a,b}$. The relevant terms are

$$\begin{aligned} \mathbf{m}^{(i,s)}_{t-k|t}(\mathbf{z}) &= \breve{\mathbf{m}}^{(i,s-1)}_{t-k|t} + \mathbf{K}^{(i,s)}_{\tau^j(t)}(\mathbf{z} - \mathbf{\Gamma}_{\theta_s} \mathbf{m}^{(i,k-s)}_{t-\theta_s|t}) \quad (44) \\ \mathbf{K}^{(i,s)}_{\tau^{j_s}(t)} &= \sum_{(l_\gamma,\theta_\gamma) \in \mathfrak{A}_t} \mathbf{W}^{(i,s-1),-}_{\tau^{j_s}(t),\tau^{l_\gamma}(t)} \mathbf{\Gamma}^\top_{\theta_\gamma} (\mathbf{S}^{(j_s,\theta_s)}_{(l_\gamma,\theta_\gamma)})^{-1} \\ \mathbf{S}^{(j_s,\theta_s)}_{(l_\gamma,\theta_\gamma)} &= \mathbf{\Gamma}_{\theta_\gamma} \mathbf{W}^{(i,s-1),-}_{\tau^{l_\gamma}(t),\tau^{j_s}(t)} \mathbf{\Gamma}^\top_{\theta_s} + \delta\begin{pmatrix} (j_s, \theta_s) \\ (l_\gamma, \theta_\gamma) \end{pmatrix} \mathbf{\Lambda}_{\theta_s} \\ \mathbf{P}^{(i,s)}_{t-k|t} &= \mathbf{P}^{(i,s-1)}_{t-k|t} - \\ &\sum_{(l_\gamma,\theta_\gamma) \in \mathfrak{A}_t} \mathbf{K}^{(i,s)}_0 \mathbf{\Gamma}_{\theta_\gamma} \mathbf{W}^{(i,s-1),-}_{\tau^{l_\gamma}(t),\tau^{j_s}(t)} \quad (45) \end{aligned}$$

$$\mathbf{W}^{(i,s),+}_{a,b} = \mathbf{W}^{(i,s-1),-}_{a,b} - \sum_{(l_\gamma,\theta_\gamma) \in \mathfrak{A}_t} \mathbf{K}^{(i,s)}_a \mathbf{\Gamma}_{\theta_\gamma} \mathbf{W}^{(i,s-1),-}_{\tau^{l_\gamma}(t),b}$$

for all pairs $a, b \in \{0, \ldots, \psi\}$ with $\mathbf{W}^{i,+}_{a,b} = \mathbf{W}^{(i,|\mathfrak{A}|),-}_{a,b}$. Now it follows that $\breve{v}^s_{t-k|t}$ is a Gaussian-sum for $k = 0, \ldots, \psi$.

Again, if $\psi = 0$ then the update algorithm proposed in Theorem 2 collapses to the algorithm proposed in Proposition 2 and only $\breve{v}_{t|t}$ needs to be computed. The algorithm outlined in Theorem 1 and Theorem 2 generalizes the Gaussian-sum PHD filter outlined in [11] and is based on the generalized Kalman filter detailed in [17].

The same pruning and merging algorithm proposed in [11] can be applied in this new case in order to aid computational efficiency.

This new algorithm is a significant extension to the original algorithm proposed in [11] since it allows one to achieve the analytic, closed-form, efficiency of the original algorithm [11] in more realistic networked scenarios. Again, the scenario considered is very general where each sensor's signal can incur an independent random delay and may arrive out-of-sequence with other signals generated by the same sensor or other sensors.

6 Conclusion

A new, generalized and closed-form, algorithm for probability hypothesis density (PHD) filtering was derived which accounts for irregular measurement sequences involving time-delays and out-of-sequence measurements from multiple sensors. The proposed algorithm extends an existing closed-form PHD filter based on Gaussian mixtures [11].

To the best of the author's knowledge, no similar work in the literature examines the problem of PHD filtering with irregular (multi-sensor out-of-sequence) measurement sequences.

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